

Equidistribution of Signs for Modular Eigenforms of Half Integral Weight

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Abstract

Let f be a cusp form of weight $k + 1/2$ and at most quadratic nebentype character whose Fourier coefficients $a(n)$ are all real. We study an equidistribution conjecture of Bruinier and Kohnen for the signs of $a(n)$. We prove this conjecture for certain subfamilies of coefficients that are accessible via the Shimura lift by using the Sato-Tate equidistribution theorem for integral weight modular forms. Firstly, an unconditional proof is given for the family $\{a(tp^2)\}_p$ where t is a squarefree number and p runs through the primes. In this case, the result is in terms of the natural density. To prove it for the family $\{a(tn^2)\}_n$ where t is a squarefree number and n runs through the squarefree numbers or all natural numbers, we assume the existence of a suitable error term for the convergence of the Sato-Tate distribution, which is weaker than one conjectured by Akiyama and Tanigawa. In this case, the results are in terms of the Dedekind-Dirichlet density.

MSC (2010): 11F37 (Forms of half-integer weight; nonholomorphic modular forms); 11F30 (Fourier coefficients of automorphic forms).

1 Introduction

The signs of the coefficients of half-integral weight modular forms have attracted some recent attention. In particular, Bruinier and Kohnen conjectured an equidistribution of signs. Using the Shimura lift and the Sato-Tate equidistribution theorem we obtain results towards this conjecture.

Throughout this paper, the notation is as follows. We fix integers $k \geq 2$ and $N \in \mathbb{N}$ with $4|N$ and we let χ be an at most quadratic Dirichlet character modulo N . We denote the space of cusp forms of weight $k + 1/2$ and level N with character χ by $S_{k+1/2}(N, \chi)$ in the sense of Shimura, as in the main theorem in [17] on p. 458. Let $f = \sum_{n \geq 1} a(n)q^n \in S_{k+1/2}(N, \chi)$. For a fixed squarefree t such that $a(t) \neq 0$, there exists a map, called the Shimura correspondence, that lifts f to a cusp form F_t of weight $2k$ and level $N/2$ with character χ^2 . We assume that F_t is a cuspidal Hecke eigenform of weight $2k$ without complex multiplication.

The question about the sign changes of coefficients of half integral weight modular forms has been asked by Bruinier and Kohnen in [4] and there it was shown that the sequence $\{a(tn^2)\}_{n \in \mathbb{N}}$ of coefficients of half integral weight modular forms has infinitely many sign changes under the additional hypothesis that a certain L -function has no zeros in the interval $(0, 1)$ (later this hypothesis

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has been removed in [9]). In addition to this partly conditional result, the authors came up with a suggestion supported by some numerical experiments. They claimed that half of the coefficients are positive among all non-zero coefficients due to observations on computations made for one example of weight $11/2$ given in [11] and for one example of weight $3/2$ given in [7] for indices up to 10^6 . This claim is also supported by 5 different examples of weight $3/2$ in computations done in the preparation process of [8] for indices up to 10^7 . One can see these numerical data on the webpage [18]. In [10], Kohnen, Lau and Wu also study the sign change problem on specific sets of integers. They establish lower bounds of the best possible order of magnitude for the number of those coefficients that have the same signs. These give an improvement on some results in [4] and [9]. Although the problem of the equidistribution of signs has been mentioned informally in [4], the conjecture was only given in [10] formally.

The aim of this paper is to give a proof of this question for $\{a(tn^2)\}_n$ for n running through the set of primes (see Theorem 5.1), through the set of squarefree positive integers (see Corollary 6.2), and through the set of all positive integers (see Corollary 7.2). For the latter two results we have to assume a certain error term for the convergence of the Sato-Tate distribution for integral weight modular forms, which is much weaker than an error term conjectured by Akiyama and Tanigawa. The latter two results are formulated in terms of Dedekind-Dirichlet density.

It should be pointed out that our techniques cannot handle the interesting question of equidistribution of signs of $\{a(n)\}_n$, when n runs through the squarefree integers.

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2 Half-integral weight modular forms and the Shimura lift

Recall that $k \geq 2$. The Shimura lift maps $S_{k+1/2}(N, \chi)$ to $S_{2k}(N/2, \chi^2)$. We summarise some of its properties that are proved in [17] and Chapter 2 of [14] in the following theorem.

Theorem 2.1 (Shimura, Niwa). *Let $t \geq 1$ be a squarefree integer and $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+1/2}(N, \chi)$ be a non-zero cusp form where $q = e^{2\pi iz}$ and $z \in \mathbb{H}$ (the upper half-plane). For a fixed squarefree t such that $a(t) \neq 0$, there exists a map, called the Shimura correspondence, that sends f to a cusp form F_t of weight $2k$ and level $N/2$ with character χ^2 . The form F_t is called the Shimura lift of f associated with t . Let $F_t(z) = \sum_{n \geq 1} A_t(n)q^n$ be its Fourier expansion. There is the direct relation between their Fourier coefficients*

$$A_t(n) := \sum_{d|n} \chi_{t,N}(d) d^{k-1} a\left(\frac{tn^2}{d^2}\right), \quad (1)$$

where $\chi_{t,N}$ denotes the character

$$\chi_{t,N}(d) := \chi(d) \left(\frac{(-1)^k N^2 t}{d} \right).$$

Furthermore if f is a Hecke eigenform for all Hecke operators T_{p^2} , then so is the Shimura lift for all Hecke operators T_p . In fact, in this case

$$F_t = a(t)F,$$

where F is a normalised Hecke eigenform independent of t . Moreover, from the Euler product formula for the Fourier coefficients of half integral weight modular forms, we have the following relation for $(m, n) = 1$

$$a(tm^2)a(tn^2) = a(t)a(tm^2n^2). \quad (2)$$

Note that the assumption that χ be (at most) quadratic implies that F_t has real coefficients if f does.

3 Density

We now recall notions about sets of prime numbers. By \mathbb{P} we always denote the set of all prime numbers.

Definition 3.1. Let S be a set of primes. The set S is said to have Dirichlet density $\delta(S)$ if the limit

$$\lim_{z \rightarrow 1^+} \frac{\sum_{p \in S} \frac{1}{p^z}}{\log \left(\frac{1}{z-1} \right)}$$

exists and is equal to $\delta(S)$. The limit $\lim_{z \rightarrow 1^+}$ is defined via any sequence of real numbers tending to 1 from the right.

For the next definition we follow [13], p. 343f.

Definition 3.2. Let S be a set of primes. We call S regular if there is $a \in \mathbb{R}$ and a function $g(z)$ which is holomorphic on $\operatorname{Re}(z) \geq 1$ such that

$$\sum_{p \in S} \frac{1}{p^z} = a \log \left(\frac{1}{z-1} \right) + g(z).$$

If S is regular, it directly follows that it has a Dirichlet density, namely $\delta(S) = a$.

Using standard properties of the Riemann zeta-function it is easy to show the following well-known property.

Lemma 3.3. The function $P(z) := \sum_{p \in \mathbb{P}} \frac{1}{p^z} - \log \left(\frac{1}{z-1} \right)$ is holomorphic on $\operatorname{Re}(z) \geq 1$.

Hence, we obtain the example that \mathbb{P} is a regular set of primes of Dirichlet density 1. It also follows that the denominator in Definition 3.1 can be replaced by $\sum_{p \in \mathbb{P}} \frac{1}{p^z}$.

We collect some properties of regular sets of primes in the following simple lemma.

Lemma 3.4. (a) Let S be any set of primes such that the series $\sum_{p \in S} \frac{1}{p}$ converges to a finite value. Then S has a Dirichlet density equal to 0.

(b) Let S be a regular set of primes. Then the following statements are equivalent:

- (i) The Dirichlet density of S is 0.
- (ii) The series $\sum_{p \in S} \frac{1}{p}$ converges to a finite value.
- (c) Let S_1, S_2 be two regular sets of primes having the same Dirichlet density $\delta(S_1) = \delta(S_2)$. Then the function $\sum_{p \in S_1} \frac{1}{p^z} - \sum_{q \in S_2} \frac{1}{q^z}$ is analytic on $\text{Re}(z) \geq 1$.

A different, maybe more intuitive notion of density, is the following one.

Definition 3.5. Let S be a set of primes. The set S is said to have natural (asymptotic) density $d(S)$ if the limit

$$\lim_{n \rightarrow \infty} \frac{\#\{p \leq x \mid p \in S\}}{\#\{p \leq x \mid p \in \mathbb{P}\}} = \lim_{n \rightarrow \infty} \frac{\pi_S(x)}{\pi(x)}$$

exists and is equal to $d(S)$. Here $\pi(x) = \#\{p \leq x \mid p \in \mathbb{P}\}$ is the prime number counting function and $\pi_S(x) := \#\{p \leq x \mid p \in S\}$.

It is well-known that a set of prime numbers S having a natural density also has a Dirichlet density, and that then the two are the same. The converse, however, fails in general (see, e.g. [16], Chapter VI, Section 4).

For our application in Section 6 we need regular sets of primes. Hence, we now include a proposition showing that if a set of primes S has a natural density and the convergence satisfies a certain error term, then it is regular. This is presumably very well-known, but, we are not aware of any reference; so we give the full proof.

Proposition 3.6. Let S be a set of primes having natural density $d(S)$. We suppose that the convergence of the natural density is ‘good enough’ in the following sense. Let $E(x) := \frac{\pi_S(x)}{\pi(x)} - d(S)$ be the error function. Suppose that there are $\alpha > 0$, $C > 0$ and $M > 0$ such that for all $x > M$ we have

$$|E(x)| \leq Cx^{-\alpha}.$$

Then S is a regular set of primes having Dirichlet density $\delta(S) = d(S)$.

Proof. We will use the notation $D_S(z) := \sum_{p \in S} \frac{1}{p^z}$ and $D(z) := \sum_{p \in \mathbb{P}} \frac{1}{p^z}$. We can rewrite $D_S(z)$ as

$$D_S(z) = \sum_{p \in \mathbb{P}} \frac{1}{p^z} = \sum_{n=2}^{\infty} \pi_S(n) \left(\frac{1}{n^z} - \frac{1}{(n+1)^z} \right) \quad (3)$$

since $\pi_S(n) - \pi_S(n-1) = \begin{cases} 1 & \text{if } n \text{ is a prime in } S, \\ 0 & \text{if } n \text{ is not a prime in } S, \end{cases}$ and similarly for $D(z)$. Let us abbreviate $d := d(S)$ and put $g(x) := E(x)\pi(x)$. Then we have for $\text{Re}(z) > 1$

$$\begin{aligned} D_S(z) &= \sum_{n=2}^{\infty} \pi_S(n) \left(\frac{1}{n^z} - \frac{1}{(n+1)^z} \right) \\ &= d \sum_{n=2}^{\infty} \pi(n) \left(\frac{1}{n^z} - \frac{1}{(n+1)^z} \right) + \sum_{n=2}^{\infty} g(n) \left(\frac{1}{n^z} - \frac{1}{(n+1)^z} \right) \\ &= dD(z) + \sum_{n=2}^{\infty} g(n) \left(\frac{1}{n^z} - \frac{1}{(n+1)^z} \right). \end{aligned}$$

Now we show that the last sum defines a function that is holomorphic for $\operatorname{Re}(z) \geq 1$, which implies the assertion of the proposition in view of Lemma 3.3.

Note that

$$\frac{1}{n^z} - \frac{1}{(n+1)^z} = z \int_n^{n+1} \frac{1}{x^{z+1}} dx.$$

Note also that $g(x)$ is a step function with jumps only at integers, thus

$$\sum_{n=2}^{\infty} g(n) \left(\frac{1}{n^z} - \frac{1}{(n+1)^z} \right) = \sum_{n=2}^{\infty} g(n) \int_n^{n+1} \frac{1}{x^{z+1}} dx = \sum_{n=2}^{\infty} \int_n^{n+1} \frac{g(x)}{x^{z+1}} dx = \int_2^{\infty} \frac{g(x)}{x^{z+1}} dx.$$

From Theorem 29 of [15], we know that $\pi(x) < \frac{x}{\log(x)-4}$ for $x > 55$. Plugging this fact into $|g(x)|$ we have

$$|g(x)| = |E(x)\pi(x)| \leq C \cdot \pi(x) \cdot x^{-\alpha} \leq C \cdot \frac{x^{1-\alpha}}{\log(x)-4}.$$

Using the last inequality we have for $\operatorname{Re}(z) > 1 - \frac{\alpha}{2}$

$$\begin{aligned} \left| \int_{55}^{\infty} \frac{g(x) dx}{x^{z+1}} \right| &\leq \int_{55}^{\infty} \frac{|g(x)|}{x^{\operatorname{Re}(z)+1}} dx \\ &\leq C \int_{55}^{\infty} \frac{1}{x^{1+\frac{\alpha}{2}} \cdot (\log(x)-4)} dx \\ &\leq C \int_{55}^{\infty} \frac{1}{x^{1+\frac{\alpha}{2}}} dx. \end{aligned}$$

Note that the last integral is convergent. We conclude that $\sum_{n=2}^{\infty} g(n) \left(\frac{1}{n^z} - \frac{1}{(n+1)^z} \right)$ defines an analytic function on $\operatorname{Re}(z) \geq 1$. \square

The error term in the proposition is the simplest one for which we could obtain the conclusion of the proposition. If necessary, it can certainly be improved a little.

We end this section by recalling a notion of density on sets of natural numbers analogous to that of Dirichlet density of sets of prime numbers. Two of our main results about the equidistribution of signs for coefficients of half-integral weight modular forms will be formulated for this type of density.

Definition 3.7. Let $A \subseteq \mathbb{N}$ be a subset. We say A has Dedekind-Dirichlet density $\delta(A)$ if the limit

$$\lim_{z \rightarrow 1^+} (z-1) \sum_{n=1, n \in A}^{\infty} \frac{1}{n^z}$$

exists and is equal to $\delta(A)$.

If $A \subseteq \mathbb{N}$ has a natural density (defined in the evident way), then by the Dedekind-Dirichlet Theorem (see [3], Theorem 1.8 on p. 10) it has a Dedekind-Dirichlet density, and they are the same.

As an example let us note that the set of square-free integers has natural and Dedekind-Dirichlet density $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$, see p. 4 of [3].

4 Sato-Tate equidistribution

Our approach to the equidistribution of signs for a half-integral weight modular form f is based on applying the Sato-Tate equidistribution to the Shimura lift of f .

We first recall some basic notions, specialised to our situation.

Definition 4.1. *The Sato-Tate measure μ is the probability measure on the interval $[-1, 1]$ given by $\frac{2}{\pi} \sqrt{1-t^2} dt$.*

A sequence $\{s_1, s_2, s_3, \dots\}$ of real numbers in $[-1, 1]$ is said to be equidistributed with respect to the Sato-Tate measure μ if for all continuous functions $f : [-1, 1] \rightarrow \mathbb{R}$ the following limit exists and satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(s_i) = \mu(f) = \int_{-1}^1 f(t) \frac{2}{\pi} \sqrt{1-t^2} dt.$$

Let $k \geq 1$ and F be a normalised cuspidal eigenform of weight $2k$, whose p -th Fourier coefficient is denoted by $A(p)$. It satisfies the well-known Ramanujan-Petersson bound

$$|A(p)| \leq 2p^{k-1/2},$$

proved by Deligne as a consequence of his proof of the Weil conjectures in [6]. We let

$$B(p) := \frac{A(p)}{2p^{k-1/2}} \in [-1, 1].$$

Theorem B of [2], case 3 with $\zeta = 1$, gives the important *Sato-Tate equidistribution theorem* for $\Gamma_0(N)$:

Theorem 4.2 (Barnet-Lamb, Geraghty, Harris, Taylor). *Let $k \geq 1$ and let $F = \sum_{n \geq 1} A(n)q^n$ be a normalised cuspidal Hecke eigenform of weight $2k$ for $\Gamma_0(N)$ without complex multiplication.*

Then the numbers $B(p)$ are equidistributed in $[-1, 1]$ with respect to the Sato-Tate measure, when p runs through the primes not dividing N .

We now include the well-known corollary breaking down the equidistribution statement into an explicit assertion about densities.

Corollary 4.3. *Assume the setup of Theorem 4.2. Let $[a, b] \subseteq [-1, 1]$ be a subinterval and let*

$$S_{[a,b]} := \{p \text{ prime} \mid p \nmid N, B(p) \in [a, b]\}.$$

Then $S_{[a,b]}$ has natural density (and hence Dirichlet density) equal to

$$\frac{2}{\pi} \int_a^b \sqrt{1-t^2} dt.$$

Proof. The assertion on the natural density is an immediate consequence of applying Theorem 4.2 to the characteristic function of the interval $[a, b]$. \square

5 Equidistribution of signs for $\{a(tp^2)\}_{p \text{ prime}}$

In this section we prove our results on the equidistribution of signs of $\{a(tp^2)\}_{p \text{ prime}}$. Our main unconditional result is the following theorem.

Theorem 5.1. *Let $k \geq 2$, $4 \mid N$ be integers, χ a Dirichlet character modulo N s.t. $\chi^2 = 1$ and let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+1/2}(N, \chi)$ be a non-zero normalised Hecke eigenform with real coefficients. Let $F_t = \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2k}(N/2, \chi^2)$ be the Hecke eigenform corresponding to f under the Shimura lift for a fixed squarefree t such that $a(t) \neq 0$, see Theorem 2.1. Furthermore we assume that F_t has no complex multiplication. Define the set of primes*

$$\mathbb{P}_{>0} := \{p \in \mathbb{P} \mid a(tp^2) > 0\}$$

and similarly $\mathbb{P}_{<0}$, $\mathbb{P}_{\geq 0}$, $\mathbb{P}_{\leq 0}$, and $\mathbb{P}_{=0}$ (depending on f and t).

Then the sets $\mathbb{P}_{>0}$, $\mathbb{P}_{<0}$, $\mathbb{P}_{\geq 0}$, $\mathbb{P}_{\leq 0}$ have natural density $1/2$ and the set $\mathbb{P}_{=0}$ has natural density 0.

Proof. Denote by $\pi_{>0}(x) := \#\{p \leq x \mid p \in \mathbb{P}_{>0}\}$ and similarly $\pi_{<0}(x)$, $\pi_{\geq 0}(x)$, $\pi_{\leq 0}(x)$, and $\pi_{=0}(x)$.

Since dividing f by $a(t)$ does not affect the assertions of the theorem, we may and do assume $a(t) = 1$. In that case F_t is a normalised eigenform (see Theorem 2.1).

We will use the relation between the coefficients of f and F_t stated in Theorem 2.1:

$$a(p^2t) = A_t(p) - \chi_{t,N}(p)p^{k-1} \quad (4)$$

for all primes p . The idea is to use the Sato-Tate equidistribution to show that $|A_t(p)|$ dominates the term $\chi_{t,N}(p)p^{k-1}$ for ‘most’ primes. Let us use the Sato-Tate normalisation $B_t(p) := \frac{A_t(p)}{2p^{k-1/2}}$ as in Theorem 4.2. Equation (4) directly implies the equivalence:

$$a(p^2t) > 0 \Leftrightarrow B_t(p) > \frac{\chi_{t,N}(p)}{2\sqrt{p}}.$$

Let $\epsilon > 0$. For all $p > \frac{1}{4\epsilon^2}$ one has

$$\left| \frac{\chi_{t,N}(p)}{2\sqrt{p}} \right| = \frac{1}{2\sqrt{p}} < \epsilon.$$

Thus we obtain

$$\pi_{>0}(x) + \pi\left(\frac{1}{4\epsilon^2}\right) \geq \#\{p \leq x \text{ prime} \mid B_t(p) > \epsilon\}. \quad (5)$$

By Corollary 4.3, we have

$$\frac{\#\{p \leq x \text{ prime} \mid B_t(p) > \epsilon\}}{\pi(x)} \xrightarrow{x \rightarrow \infty} \mu([\epsilon, 1]),$$

where μ is the Sato-Tate measure on $[-1, 1]$. This implies that

$$\liminf_{x \rightarrow \infty} \frac{\pi_{>0}(x)}{\pi(x)} \geq \mu([\epsilon, 1])$$

for all $\epsilon > 0$, whence

$$\liminf_{x \rightarrow \infty} \frac{\pi_{>0}(x)}{\pi(x)} \geq \mu([0, 1]) = \frac{1}{2}.$$

A similar argument yields

$$\liminf_{x \rightarrow \infty} \frac{\pi_{\leq 0}(x)}{\pi(x)} \geq \mu([0, 1]) = \frac{1}{2}.$$

Using $\pi_{\leq 0}(x) = \pi(x) - \pi_{> 0}(x)$ gives

$$\limsup_{x \rightarrow \infty} \frac{\pi_{> 0}(x)}{\pi(x)} \leq \mu([0, 1]) = \frac{1}{2},$$

thus showing that $\lim_{x \rightarrow \infty} \frac{\pi_{> 0}(x)}{\pi(x)}$ exists and is equal to $\frac{1}{2}$, whence by definition $\mathbb{P}_{> 0}$ has natural density $\frac{1}{2}$. The arguments for $\mathbb{P}_{< 0}(x)$, $\mathbb{P}_{\geq 0}(x)$, $\mathbb{P}_{\leq 0}(x)$ are exactly the same, and the conclusion for $\mathbb{P}_{=0}$ immediately follows. \square

We next show that the sets of primes in Theorem 5.1 are regular if the Sato-Tate equidistribution converges with a certain error term. A much stronger error term was conjectured by Akiyama-Tanigawa (see [12], Conjecture 2.2, and Conjecture 1, on p. 1204 in [1]).

Theorem 5.2. *We make the same assumptions as in Theorem 5.1. We additionally assume that there are $C > 0$ and $\alpha > 0$ such that for all subintervals $[a, b] \in [-1, 1]$ one has*

$$\left| \frac{\#\{p \leq x \text{ prime} \mid \frac{A_t(p)}{a(t)2p^{k-1/2}} \in [a, b]\}}{\pi(x)} - \mu([a, b]) \right| \leq \frac{C}{x^\alpha}.$$

Then the sets $\mathbb{P}_{> 0}$, $\mathbb{P}_{< 0}$, $\mathbb{P}_{\geq 0}$, $\mathbb{P}_{\leq 0}$, and $\mathbb{P}_{=0}$ are regular sets of primes.

Proof. We start as in the proof of Theorem 5.1 up to Equation (5) and plug in the error term to get

$$\begin{aligned} \pi_{> 0}(x) + \pi\left(\frac{1}{4\epsilon^2}\right) &\geq \#\{p \leq x \text{ prime} \mid B_t(p) > \epsilon\} \\ &\geq -C\pi(x)x^{-\alpha} + \mu([\epsilon, 1])\pi(x) \\ &\geq -C\pi(x)x^{-\alpha} + \left(\frac{1}{2} - \mu([0, \epsilon])\right)\pi(x). \end{aligned}$$

Rearranging the terms and dividing by $\pi(x)$ we obtain

$$\frac{\pi_{> 0}(x)}{\pi(x)} - \frac{1}{2} \geq -(Cx^{-\alpha} + \mu([0, \epsilon]) + \frac{\pi(\frac{1}{4\epsilon^2})}{\pi(x)}),$$

which is valid for all $\epsilon > 0$ and all $x > 0$. Using the estimates $\mu([0, \epsilon]) = \int_0^\epsilon \sqrt{1-t^2} dt \leq \epsilon$ and $\frac{x}{\log(x)+2} \leq \pi(x) \leq \frac{x}{\log(x)-4}$, which is valid for $x > 55$ (see [15] Theorem 29 A on p. 211), we get

$$\frac{\pi_{> 0}(x)}{\pi(x)} - \frac{1}{2} \geq -(Cx^{-\alpha} + \epsilon + \frac{(\log(x)+2)}{-4x\epsilon^2(\log(4\epsilon^2)+4)}),$$

valid for all $\epsilon > 0$ and all $x > 55$. Setting for instance $\epsilon = x^{-2\alpha}$, one deduces that there is a $C_1 > 0$ such that for big enough x one has

$$\frac{\pi_{> 0}(x)}{\pi(x)} - \frac{1}{2} \geq -C_1 x^{-\alpha}.$$

We now make the same argument with $\pi_{<0}(x)$ to obtain

$$\frac{\pi_{<0}(x)}{\pi(x)} - \frac{1}{2} \geq -C_1 x^{-\alpha}.$$

Using $\pi_{\geq 0}(x) = \pi(x) - \pi_{<0}(x)$, we get

$$\frac{\pi_{\geq 0}(x)}{\pi(x)} - \frac{1}{2} \leq \frac{\pi_{\geq 0}(x)}{\pi(x)} - \frac{1}{2} \leq C_1 x^{-\alpha}.$$

Thus, one has

$$\left| \frac{\pi_{>0}(x)}{\pi(x)} - \frac{1}{2} \right| \leq \frac{C_1}{x^\alpha}.$$

Proposition 3.6 now implies that $\mathbb{P}_{>0}$ is a regular set of primes. The regularity of $\mathbb{P}_{<0}$ is obtained in the same way, implying also the regularity of $\mathbb{P}_{=0}$, $\mathbb{P}_{\geq 0}$ and $\mathbb{P}_{\leq 0}$. \square

The form of the error term in Theorem 5.2, of course, comes from Proposition 3.6.

Remark 5.3. *The same arguments as in Theorem 5.1 can be used to prove the following assertion:*

Assume the setup of Theorem 5.1. Let $[a, b] \subseteq [-1, 1]$ be a subinterval. Then the set of primes $\{p \mid \frac{a(tp^2)}{2a(t)p^{k-1/2}} \in [a, b]\}$ has a natural density equal to $\mu([a, b])$.

Similarly, a more general version of Theorem 5.2 also holds.

6 Equidistribution of signs for $\{a(tn^2)\}_{n \text{ squarefree}}$

Under the assumption of a certain error term for the convergence of the Sato-Tate distribution, we obtain the following equidistribution result for the signs of the coefficients $a(tn^2)$, when n runs through the squarefree integers.

Theorem 6.1. *We make the same assumptions as in Theorem 5.1. As in Theorem 5.2 we additionally assume that there are $C > 0$ and $\alpha > 0$ such that for all subintervals $[a, b] \in [-1, 1]$ one has*

$$\left| \frac{\#\{p \leq x \text{ prime} \mid \frac{A_t(p)}{a(t)2p^{k-1/2}} \in [a, b]\}}{\pi(x)} - \mu([a, b]) \right| \leq \frac{C}{x^\alpha}.$$

We also assume $a(t) > 0$ for simplicity (in the opposite case, some signs have to be inverted). We define the multiplicative (but, not completely multiplicative) function

$$s(n) = \begin{cases} 0 & \text{if } n \text{ is not squarefree} \\ 1 & \text{if } n \text{ is squarefree and } a(tn^2) > 0, \\ -1 & \text{if } n \text{ is squarefree and } a(tn^2) < 0, \\ 0 & \text{if } n \text{ is squarefree and } a(tn^2) = 0. \end{cases}$$

Let $S(z) := \sum_{n=1}^{\infty} \frac{s(n)}{n^z}$ be the Dirichlet series of $s(n)$.

Then $S(z)$ is holomorphic for $\operatorname{Re}(z) \geq 1$.

Proof. Since $s(n)$ is multiplicative because of Equation (2) and the assumption $a(t) > 0$, we can use the fact that $S(z) = \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} s(p^k) p^{-kz}$ due to Theorem 1.4.1 on p. 24 in [5]. Using these facts with the definition of $s(n)$ we conclude that the Euler product of $S(z)$ is

$$S(z) = \prod_{p \in \mathbb{P}_{>0}} \left(1 + \frac{1}{p^z}\right) \prod_{p \in \mathbb{P}_{<0}} \left(1 - \frac{1}{p^z}\right).$$

We take the logarithm on both sides:

$$\log S(z) = \sum_{p \in \mathbb{P}_{>0}} \log\left(1 + \frac{1}{p^z}\right) + \sum_{p \in \mathbb{P}_{<0}} \log\left(1 - \frac{1}{p^z}\right).$$

Now expanding out the logarithm we are precisely led to consider the function

$$\sum_{p \in \mathbb{P}_{>0}} \frac{1}{p^z} - \sum_{p \in \mathbb{P}_{<0}} \frac{1}{p^z}.$$

Since $\mathbb{P}_{>0}$ and $\mathbb{P}_{<0}$ are regular sets of primes having the same density $\frac{1}{2}$ by Theorem 5.2, we can conclude that $\log S(z)$ is analytic for $\operatorname{Re}(z) \geq 1$ by Lemma 3.4. Since the exponential function is holomorphic, we have $S(z)$ is analytic for $\operatorname{Re}(z) \geq 1$ by taking the exponential of $\log S(z)$, as was to be shown. \square

We now translate Theorem 6.1 into a density statement.

Corollary 6.2. *Assume the setting of Theorem 6.1. The sets*

$$\{n \in \mathbb{N} \mid n \text{ squarefree and } a(tn^2) > 0\} \text{ and } \{n \in \mathbb{N} \mid n \text{ squarefree and } a(tn^2) < 0\}$$

have equal positive Dedekind-Dirichlet densities, that is, both are precisely half of the density of the set of

$$\{n \in \mathbb{N} \mid n \text{ squarefree and } a(tn^2) \neq 0\}.$$

Proof. By definition of $s(n)$ we have

$$S(z) = \sum_{n \text{ squarefree, } a(tn^2) > 0} \frac{1}{n^z} - \sum_{n \text{ squarefree, } a(tn^2) < 0} \frac{1}{n^z}.$$

Since the set of all squarefree numbers has a Dedekind-Dirichlet density of $\frac{1}{\zeta(2)}$, we can write

$$\lim_{z \rightarrow 1^+} (z-1) \sum_{n \text{ squarefree}} \frac{1}{n^z} = \frac{1}{\zeta(2)}.$$

Since $S(z)$ is holomorphic for $\operatorname{Re}(z) \geq 1$ by Theorem 6.1, we can write

$$\lim_{z \rightarrow 1^+} (z-1) \left[2 \sum_{n \text{ squarefree, } a(tn^2) > 0} \frac{1}{n^z} + \sum_{n \text{ squarefree, } a(tn^2) = 0} \frac{1}{n^z} \right] = \frac{1}{\zeta(2)}.$$

If we show that $\sum_{n \text{ squarefree, } a(tn^2) = 0} \frac{1}{n^z}$ is analytic for $\operatorname{Re}(z) \geq 1$ then we will get the desired result. To do this define $t(n) := [s(n)]^2$ and $T(z) := \sum_n \frac{t(n)}{n^z}$. Since $t(n)$ is multiplicative, we can use the Euler product of $T(z)$:

$$T(z) = \sum_{n=1}^{\infty} \frac{t(n)}{n^z} = \prod_{p, a(tp^2) \neq 0} \left(1 + \frac{1}{p^z}\right).$$

It is well known that

$$g(z) := \sum_{n \text{ squarefree}} \frac{1}{n^z} = \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^z}\right).$$

Taking the logarithm of $A(z) := \frac{T(z)}{g(z)}$ and expanding out of the logarithm we have

$$\log A(z) = \log \frac{\prod_{p, a(tp^2) \neq 0} \left(1 + \frac{1}{p^z}\right)}{\prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^z}\right)} = - \sum_{p, a(tp^2)=0} \frac{1}{p^z} + k(z)$$

where $k(z)$ is a holomorphic function for $\operatorname{Re}(z) \geq 1$. We know that $\sum_{p, a(tp^2)=0} \frac{1}{p^z}$ is holomorphic for $\operatorname{Re}(z) \geq 1$, since $\mathbb{P}_{=0}$ is a regular set of primes of density 0 by Theorem 5.2. Taking the exponential shows that $A(z) = \frac{T(z)}{g(z)}$ is holomorphic for $\operatorname{Re}(z) \geq 1$ and that $A(1) > 0$. Taking the limit for $z \rightarrow 1^+$, we obtain

$$\lim_{z \rightarrow 1^+} (z-1)T(z) = \frac{A(1)}{\zeta(2)}.$$

Therefore we conclude that the Dedekind-Dirichlet density of the set

$$\{n \in \mathbb{N} \mid n \text{ squarefree and } a(n^2) \neq 0\}$$

is equal to $\frac{A(1)}{\zeta(2)}$. Hence

$$\lim_{z \rightarrow 1} (z-1) (g(z) - T(z)) = \lim_{z \rightarrow 1^+} (z-1) \sum_{n \text{ squarefree}, a(n^2)=0} \frac{1}{n^z} = \frac{1-A(1)}{\zeta(2)}.$$

So we conclude that

$$\lim_{z \rightarrow 1^+} (z-1) \sum_{n \text{ squarefree}, a(n^2)>0} \frac{1}{n^z} = \frac{A(1)}{2\zeta(2)}.$$

This implies that the Dedekind-Dirichlet densities of the two sets in the statement are equal and this completes the proof. \square

7 Equidistribution of signs for $\{a(n^2)\}_{n \in \mathbb{N}}$

In this section we show that the arguments (and assumptions) of Section 6 can be made to work to yield an equidistribution result for the signs of the coefficients $a(n^2)$, when n runs through the natural numbers.

Theorem 7.1. *We make the same assumptions as in Theorem 5.1. As in Theorem 5.2 we additionally assume that there are $C > 0$ and $\alpha > 0$ such that for all subintervals $[a, b] \in [-1, 1]$ one has*

$$\left| \frac{\#\{p \leq x \text{ prime} \mid \frac{A_t(p)}{a(t)2p^{k-1/2}} \in [a, b]\}}{\pi(x)} - \mu([a, b]) \right| \leq \frac{C}{x^\alpha}.$$

We also assume $a(t) > 0$ for simplicity (in the opposite case, some signs have to be inverted). We define the multiplicative (but, not completely multiplicative) function

$$s(n) = \begin{cases} 1 & \text{if } a(n^2) > 0, \\ -1 & \text{if } a(n^2) < 0, \\ 0 & \text{if } a(n^2) = 0. \end{cases}$$

Let $S(z) := \sum_{n=1}^{\infty} \frac{s(n)}{n^z}$ be the Dirichlet series of $s(n)$.

Then $S(z)$ is holomorphic for $\operatorname{Re}(z) \geq 1$.

Proof. Since $s(n)$ is multiplicative because of Equation 2 and the assumption $a(t) > 0$, we can use the fact that $S(z) = \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} s(p^k) p^{-kz}$ due to Theorem 1.4.1 on p. 24 in [5]. Using these facts with the definition of $s(n)$ we conclude that the Euler product of $S(z)$ is

$$\prod_{p \in \mathbb{P}_{>0}} \left(1 + \frac{1}{p^z} + \frac{s(p^2)}{p^{2z}} + \frac{s(p^3)}{p^{3z}} + \dots\right) \prod_{p \in \mathbb{P}_{<0}} \left(1 - \frac{1}{p^z} + \frac{s(p^2)}{p^{2z}} + \frac{s(p^3)}{p^{3z}} \dots\right) \prod_{p \in \mathbb{P}_{=0}} \left(1 + \frac{s(p^2)}{p^{2z}} + \frac{s(p^3)}{p^{3z}} \dots\right).$$

Taking logarithm of $S(z)$, we have

$$\log S(z) = \sum_{p \in \mathbb{P}_{>0}} \log \left(1 + \frac{1}{p^z} + g(z, p)\right) + \sum_{p \in \mathbb{P}_{<0}} \log \left(1 - \frac{1}{p^z} + g(z, p)\right) + \sum_{p \in \mathbb{P}_{=0}} \log (1 + g(z, p)),$$

where $g(z, p) := \sum_{m=2}^{\infty} \frac{s(p^m)}{p^{mz}}$ is a holomorphic function on $\operatorname{Re}(z) \geq 1$ in the variable z for fixed p . Now expanding out the logarithm, we have,

$$\begin{aligned} \log S(z) &= \sum_{p \in \mathbb{P}_{>0}} \left(\frac{1}{p^z} + g(z, p) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{1}{p^z} + g(z, p) \right)^n \right) \\ &\quad + \sum_{p \in \mathbb{P}_{<0}} \left(-\frac{1}{p^z} + g(z, p) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \left(-\frac{1}{p^z} + g(z, p) \right)^n \right) \\ &\quad + \sum_{p \in \mathbb{P}_{=0}} \left(g(z, p) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} (g(z, p))^n \right). \end{aligned}$$

Note that for all p , we have

$$|g(z, p)| \leq \frac{1}{p^z(p^z - 1)}.$$

Using this fact, we conclude that

$$\log S(z) \leq \sum_{p \in \mathbb{P}_{>0}} \frac{1}{p^z} - \sum_{p \in \mathbb{P}_{<0}} \frac{1}{p^z} + k(z)$$

where $k(z)$ is holomorphic function for $\operatorname{Re}(z) \geq 1$.

Since $\mathbb{P}_{>0}$ and $\mathbb{P}_{<0}$ are regular set of primes having the same density $\frac{1}{2}$ by Theorem 5.2, we can conclude that $\log S(z)$ is analytic for $\operatorname{Re}(z) \geq 1$ by Lemma 3.4. Since the exponential function is holomorphic, we find that $S(z)$ is analytic for $\operatorname{Re}(z) \geq 1$ by taking the exponential of $\log S(z)$, as was to be shown. \square

We now translate Theorem 7.1 into a density statement.

Corollary 7.2. *Assume the setting of Theorem 7.1. The sets*

$$\{n \in \mathbb{N} \mid a(tn^2) > 0\} \text{ and } \{n \in \mathbb{N} \mid a(tn^2) < 0\}$$

have equal positive Dedekind-Dirichlet densities, that is, both are precisely half of the density of the set of

$$\{n \in \mathbb{N} \mid a(tn^2) \neq 0\}.$$

Proof. By definition of $s(n)$ we have

$$S(z) = \sum_{a(tn^2) > 0} \frac{1}{n^z} - \sum_{a(tn^2) < 0} \frac{1}{n^z}.$$

Since the set of all natural numbers has a Dedekind-Dirichlet density of 1, we can write

$$\lim_{z \rightarrow 1^+} (z-1) \sum_{n \in \mathbb{N}} \frac{1}{n^z} = 1.$$

Since $S(z)$ is holomorphic for $\operatorname{Re}(z) \geq 1$ by Theorem 7.1, we can write

$$\lim_{z \rightarrow 1^+} (z-1) \left[2 \sum_{a(tn^2) > 0} \frac{1}{n^z} + \sum_{a(tn^2) = 0} \frac{1}{n^z} \right] = 1.$$

To get the desired result, it is sufficient to show that $\sum_{a(tn^2) = 0} \frac{1}{n^z}$ is analytic for $\operatorname{Re}(z) \geq 1$. Define $t(n) := [s(n)]^2$ and $T(z) := \sum_{n=1}^{\infty} \frac{t(n)}{n^z}$. Since $t(n)$ is multiplicative, we can use the Euler product of $T(z)$:

$$T(z) = \sum_{n=1}^{\infty} \frac{t(n)}{n^z} = \prod_p \left(1 + \sum_{n=1, a(tp^{2n}) \neq 0}^{\infty} \frac{1}{p^{nz}} \right).$$

Put $A(z) := \frac{T(z)}{\zeta(z)}$, then

$$\begin{aligned} A(z) &= \prod_p \left(1 - \frac{1}{p^z} \right) \cdot \prod_p \left(1 + \sum_{n=1, a(tp^{2n}) \neq 0}^{\infty} \frac{1}{p^{nz}} \right) \\ &= \prod_{p, a(tp^2) \neq 0} \left(1 - \frac{1}{p^z} \right) \left(1 + \frac{1}{p^z} + \sum_{n=2, a(tp^{2n}) \neq 0}^{\infty} \frac{1}{p^{nz}} \right) \\ &\quad \cdot \prod_{p, a(tp^2) = 0} \left(1 - \frac{1}{p^z} \right) \left(1 + \sum_{n=2}^{\infty} \frac{1}{p^{nz}} \right) \\ &= \prod_{p, a(tp^2) \neq 0} \left(1 - \frac{1}{p^{2z}} + r_1(z, p) \right) \cdot \prod_{p, a(tp^2) = 0} \left(1 - \frac{1}{p^z} + r_2(z, p) \right) \end{aligned}$$

where $r_1(z, p)$ and $r_2(z, p)$ are the remaining terms.

Taking the logarithm of $A(z)$, we conclude that $\sum_{p, a(tp^2) = 0} \log \left(1 - \frac{1}{p^z} + r_2(z, p) \right)$ is holomorphic for $\operatorname{Re}(z) \geq 1$, since $\mathbb{P}_{=0}$ is a regular set of primes of density 0, by Theorem 5.2. Moreover $\sum_{p, a(tp^2) \neq 0} \log \left(1 - \frac{1}{p^{2z}} + r_1(z, p) \right)$ is also holomorphic for $\operatorname{Re}(z) \geq 1$. Taking exponential shows that $A(z) = \frac{T(z)}{\zeta(z)}$ is holomorphic for $\operatorname{Re}(z) \geq 1$. We have $A(1) > 0$ and, by taking the limit for $z \rightarrow 1^+$,

$$\lim_{z \rightarrow 1^+} (z-1)T(z) = A(1).$$

Therefore we conclude that the Dedekind-Dirichlet density of the set

$$\{n \in \mathbb{N} \mid a(tn^2) \neq 0\}$$

is equal to $A(1)$. Hence

$$\lim_{z \rightarrow 1} (z-1) (\zeta(z) - T(z)) = \lim_{z \rightarrow 1^+} (z-1) \sum_{a(tn^2) = 0} \frac{1}{n^z} = 1 - A(1).$$

So we conclude that

$$\lim_{z \rightarrow 1^+} (z - 1) \sum_{a(tn^2) > 0} \frac{1}{n^z} = \frac{A(1)}{2}.$$

This implies that the Dedekind-Dirichlet density of the two sets in the statement are equal and this completes the proof. \square

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